Plan du cours

1. Introduction
   - Contenu du cours
2. Logique mathématique
   - Calcul propositionnel
   - Calcul des prédicats
   - Logique floue et aide à la décision
3. Induction
4. Analyse d’algorithmes
   - Comparaison asymptotique de fonctions
   - Complexité
5. Récurrence
6. Mathématique de la gestion
   - Théorie des graphes
   - Optimisation

Programmation linéaire

- Optimisation sous contraintes.
- Technique utilisée par de nombreuses entreprises et organisations.
- Utilisée notamment par : banques, éducation, foresterie, industrie pétrolière, transport, sidérurgie, compagnies aériennes, …

- Un premier exemple
- Solution graphique
- Applications typiques – Formulation

Un premier exemple

- Giapetto’s Woodcarving, Inc., fabrique deux types de jouets en bois : des **soldats** et des **trains**.
- Chaque soldat fabriqué se vend €27 et requiert un dépense de €10 en matières premières. De plus, chaque soldat produit génère €14 de coûts supplémentaires (salaires et frais généraux).
- Un train se vend €21 et coûte €9 en matières premières, plus €10 en coûts supplémentaires .

Un premier exemple

- La fabrication des soldats et des trains demande deux types de main-d'œuvre : **menuiserie** et **finition**.
- Un soldat demande 2 heures de travail de finition et 1 heure de menuiserie .
- Un train demande 1 heure de travail de finition et 1 heure de menuiserie .

Un premier exemple

- Hebdomadairement, Giapetto peut disposer de toutes les matières premières nécessaires à la fabrication mais l'entreprise ne dispose **que** de 100 heures de finition et 80 heures de menuiserie .
- La **demande** pour les trains est illimitée, mais un maximum de 40 soldats peuvent être vendus chaque semaine .
- Giapetto souhaite **maximiser son profit hebdomadaire** (revenus - coûts).

Formulation (1)

- Variables de décision :
  - x_1 = nombre de soldats produits chaque sem.
  - x_2 = nombre de trains produits chaque sem.
- Fonction économique (objectif) :
  - Revenus hebdomadaires = $27x_1 + 21x_2$
  - Coûts hebdomadaires = $10x_1 + 9x_2$
  - Autres coûts variables = $14x_1 + 10x_2$
  - Fonction économique à maximiser =

$$z = 3x_1 + 2x_2$$

Formulation (2)

- Contraintes :
  - Finition (100h) : $2x_1 + x_2 \leq 100$
  - Menuiserie (80h): $x_1 + x_2 \leq 80$
  - Soldats (max 40): $x_1 \leq 40$
- Restrictions de signe :

$$x_1 \geq 0 \qquad x_2 \geq 0$$

Programme linéaire (PL - LP)

$$\text{Max } z = 3x_1 + 2x_2$$

$$2x_1 + x_2 \leq 100$$
$$x_1 + x_2 \leq 80$$
$$x_1 \leq 40$$
$$x_1 \geq 0 \qquad x_2 \geq 0$$

Hypothèses

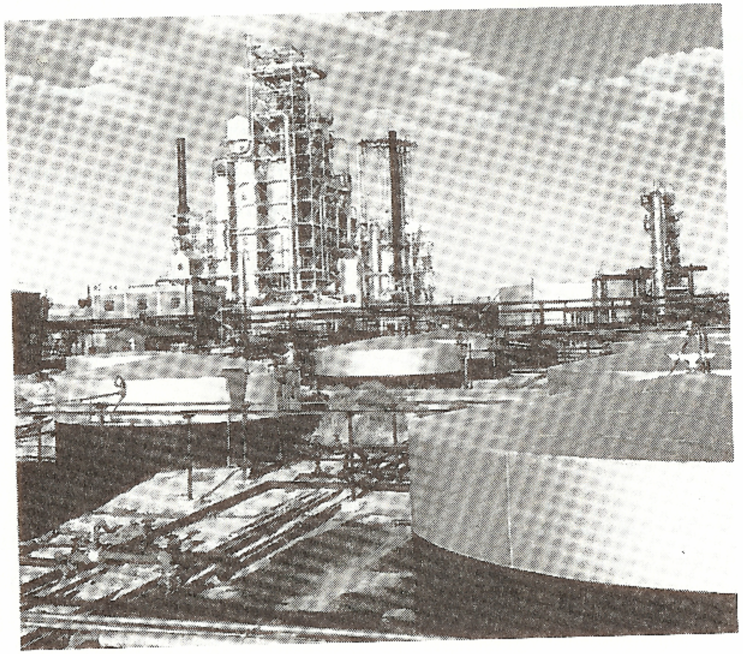
- **Proportionnalité**: Les contributions de chaque variable à la fonction économique et aux contraintes sont proportionnelles à la valeur prise par cette variable.
- **Additivité**: Les contributions de chaque variable à la fonction économique et aux contraintes sont indépendantes des valeurs prises par les autres variables.
- **Divisibilité**: Les variables de décision peuvent prendre des valeurs fractionnaires. (cf. IP)
- **Certitude**: Chaque paramètre est connu avec certitude.

Quelques exemples réels…

Management Science Application

A Refinery Linear Programming System at Citgo Petroleum

In 1983 Southland Corporation, the parent company of the 7-Eleven convenience store chain, acquired Citgo Petroleum Corporation. Prior to this acquisition, Citgo had lost money for several years; thus, a primary objective of Southland was to improve Citgo's profitability. The Southland Corporation invested in a number of management science applications to achieve this objective, and one of the largest and most important was a refinery linear programming system. The linear programming system allowed for effective management of crude stock acquisition, processing costs, and energy costs, which were almost $4 billion in 1984. The refinery linear programming system is used routinely to make decisions regarding crude selection and acquisition, refinery run levels, feedstock acquisitions, unit turnaround options, and hydrocracker conversion. The linear programming system is now one of the primary corporate operational planning tools. In 1985 Citgo achieved a pretax profit of over $70 million, and the linear programming system was cited as a significant contributor to this turnaround.



Source: D. Klingman et al., "The Successful Deployment of Management Science Throughout Citgo Petroleum Corporation," *Interfaces* 17, no. 1 (January–February 1987): 4–25.

Grape Juice Management at Welch's

With annual sales over $550 million, Welch's is one of the world's largest grape-processing companies. Founded in 1869 by Dr. Thomas B. Welch, it processes raw grapes (nearly 300,000 tons per year) into juice, as well as jellies and frozen concentrates. Welch's is owned by the National Grape Cooperative Association (NGCA), which has a membership of 1,400 growers. Welch's is NGCA's production, distribution, and marketing organization. Welch's operates its grape-processing plants near its growers. Because of the dynamic nature of product demand and customer service, Welch's holds finished goods inventory as a buffer, and maintains a large raw materials inventory stored as grape juice in refrigerated tank farms. Packaging operations at each plant draw juice from the tank farms during the year as needed. The value of the stored grape juice often exceeds $50 million. Harvest yields and grape quality vary between localities. In order to maintain national quality and consistency in its products, Welch's transfers juices between plants and adjusts product recipes. To do this Welch's uses a spreadsheet-based linear programming model. The juice logistics model (JLM) encompasses 324 decision variables and 361 constraints, that minimizes the combined costs of transporting grape juice between plants and the product recipes at each plant, and the carrying cost of storing grape juice. The model decision variables include the grape juice shipped to customers for different product groups, the grape juice transferred between plants, and inventory levels at each plant. Constraints are for recipe requirements, inventories, and grape juice usage and transfers. During the first year the linear programming model was used, it saved Welch's between $130 thousand and $170 thousand in carrying costs alone by showing Welch's it did not need to purchase extra grapes that were then available. The model has enabled Welch's management to make quick decisions regarding inventories, purchasing grapes, and adjusting product recipes when grape harvests are higher or lower than expected, and when demand changes, resulting in significant cost savings.

Source: E. W. Schuster and S. J. Allen, "Raw Material Management at Welch's, Inc.," *Interfaces* 28, no. 5 (September–October 1998): 13–24.

Selecting Freight Carriers at Reynolds Metals Company

In 1991, Reynolds Metals Company spent over $0.25 billion to deliver its products and receive raw materials through a transportation network that included truck, rail, ocean, and air shipments from over 120 shipping locations to over 5,000 receiving destinations. Truck shipments accounted for over half the company's freight costs, and interstate truck shipments alone cost over $80 million.

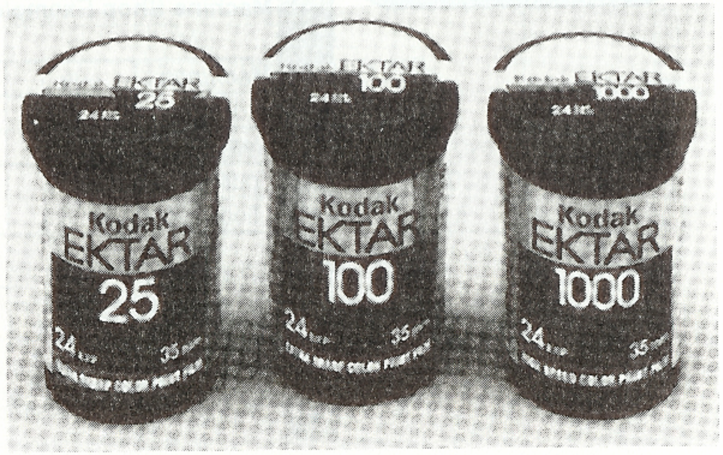
Reynolds Metals encompasses 12 decentralized operating divisions; and consistent with their decentralized operating philosophy, each division and plant traditionally was responsible for negotiating with and selecting its own carriers and arranging for shipments.

However, because of concerns about variability in service quality and high costs, in 1987 the company developed a new central dispatch system located in Richmond, Virginia, to select all (independent) truck carriers and dispatch them centrally. A vital component of this new central dispatch was a mixed integer programming model that selected the number and type of carrier to be further evaluated for final selection with a simulation model. The objective of the model was to optimize central dispatch freight costs. The model optimally selected a specific number of truck carriers and assigned them to shipping locations. Constraints included a limit on the number of carriers to be selected, carrier limits on the number of trucks they would provide Reynolds, and limits on the number of carriers allowed to service individual locations. A typical model had over 5,000 constraints, 200 integer variables, and 9,000 total variables. Based on the model results, the number of truck carriers used by Reynolds was reduced from over 200 to 14. Savings in freight costs using the entire central dispatch system is over $7 million annually, and on-time delivery service was increased from 80% to 95%.

Source: E. W. Moore Jr., "The Indispensable Role of Management Science in Centralizing Freight Operations at Reynolds Metals Company," *Interfaces* 21, no. 1 (January–February 1991): 107–29.

Minimizing Color Photographic Paper Waste at Kodak

Kodak (Australasia) Pty. Ltd., a division of Eastman Kodak Company, produces rolls of photographic color paper used to produce color photographs. In producing these rolls, the company loses some of the color paper as waste or "trim loss" in the cutting process, which results in a significant cost. The paper is originally purchased in large bulk rolls from 42 to 52.5 inches wide and up to 8,750 feet in length. The paper also comes in three different blends (i.e., different chemical coatings on the paper that react differently to light). Customers, such as one-hour photo-processing shops, order rolls from 3.5 to 11 inches wide and in lengths from 275 to 1,150 feet. To produce the customer rolls from its bulk rolls, the company must cut an entire bulk roll according to a specific design for length and width, which is referred to as a cutting plan. In order to minimize the amount of waste created when these bulk rolls are cut, Kodak developed a system of mixed integer and 0–1 programming models to determine the cutting plans for customer orders. The variables for the mixed integer model included the number of customer order lots and the number of bulk rolls used for a cutting plan, while the 0–1 integer model variables were for the paper blend for the cutting plan used. General benefits of the models included a reduction in trim waste, an increase in productivity, a reduction in the planning effort for diagramming cutting plans, and a reduced planning horizon; in addition, the company was able to better match production to customer orders. The savings in the reduction of waste alone was over $2 million in the first year of operation since waste levels were reduced by 50%.



Source: A. A. Farley, "Planning the Cutting of Photographic Color Paper Rolls for Kodak (Australasia) Pty. Ltd.," *Interfaces* 21, no. 1 (January–February 1991): 92–106.

Optimal Assignment of Gymnasts to Events

An integer programming model was developed to assign members of a women's gymnastics team to the four events conducted at a typical NCAA meet—vault, uneven bars, balance beam, and floor exercises. Each team can enter up to six gymnasts in each event, and the top five scores among these entrants contribute to the team score. At least four of the entrants must participate in all four events. These conditions formed the model constraints; the objective was to maximize the team's overall expected score. The model was tested at Utah State University and allowed officials to analyze the effects of changing conditions, such as improved performance or injuries, on the team score; to indicate to a team member why she was not selected for a particular event; and to eliminate the time the coach spent manually evaluating different team combinations.

Source: P. Ellis and R. Corn, "Using Bivalent Integer Programming to Select Teams for Intercollegiate Women's Gymnastics Competition," *Interfaces* 14, no. 3 (May–June 1984): 41–46.

Fleeting the Schedule at Delta Airlines

Each day, Delta Airlines flies over 2,500 flight legs in the United States, Canada, and Mexico. Delta uses about 450 different aircraft for these flights, arranged into 10 groups or fleets. The pattern of routes that these aircraft fly through the airline's system is the schedule, which is a crucial aspect of an airline's profitability. The schedule must be designed to maximize revenue potential by eliminating empty seats while simultaneously minimizing operating costs, which have historically been high throughout the aircraft industry. A small change in the daily system flight schedule can result in millions of dollars in revenues or losses. As a result, an airline like Delta is continuously refining and attempting to improve its schedule.

Each Delta fleet is made up of different types of aircraft, and the assignment of a particular set of aircraft to specific markets is called *fleeting the schedule.* In 1991 Delta began the Coldstart project to address the problem of fleeting its schedule. The primary goal of the fleeting process is to have the right plane at the right place at the right time; otherwise, it either will not be able to accommodate customer demand or will fly with empty seats. The results of the Coldstart project was a mixed integer linear programming model that assigns fleet types to flight legs to minimize operating and spill costs, where spill is the number of passengers not carried because of insufficient aircraft capacity.

Model operational constraints include the number of aircraft available in each fleet and scheduling requirements. The model is run daily. It normally contains about 40,000 constraints as well as 60,000 variables, about 40,000 of which are integer. The Coldstart model was implemented in September 1992, and by the summer of 1993 it was saving Delta approximately $220,000 per day. The model was expected to save Delta approximately $200 million during its first three years.

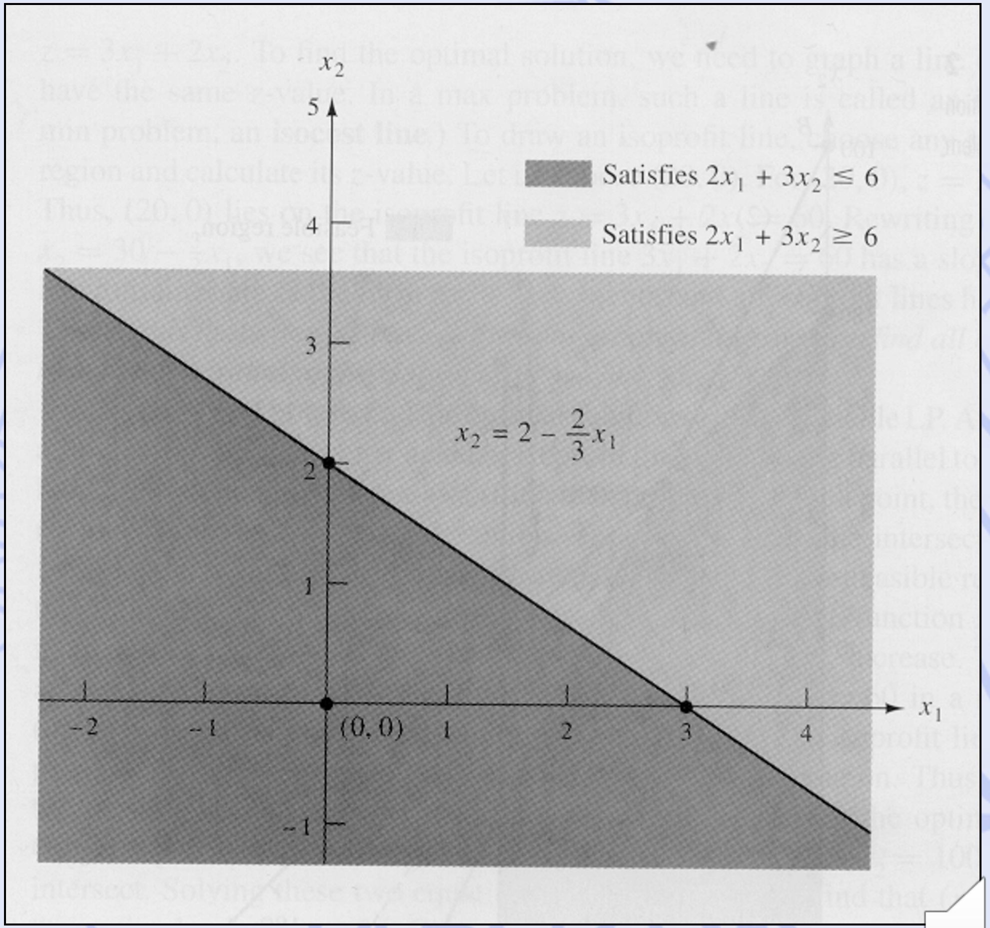
Source: R. Subramanian, et al., "Coldstart: Fleet Assignment at Delta Airlines," *Interfaces* 24, no. 1 (January–February 1994), 104–20.

Définitions

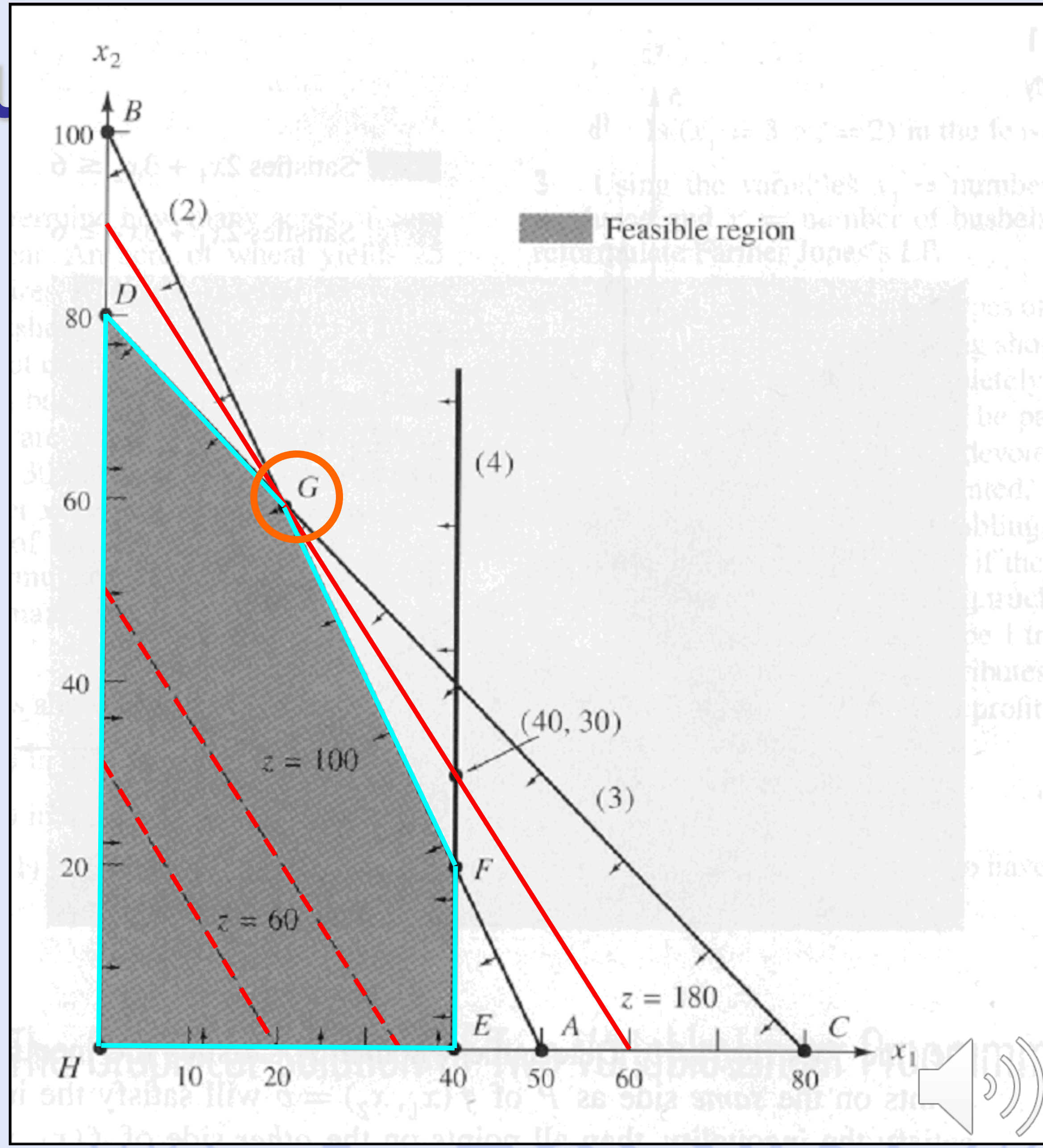
- Domaine réalisable :
  - Ensemble de tous les jeux de valeurs des variables de décision satisfaisant toutes les contraintes et restrictions de signe du PL (solutions réalisables ou solutions admissibles).
- Solution optimale :
  - Solution réalisable qui optimise (max ou min) la fonction économique.
  - Existence ?
  - Unicité ?

Solution graphique - cas de 2 variables

- Domaine réalisable = région dans le plan.
- Contrainte : équation = droite



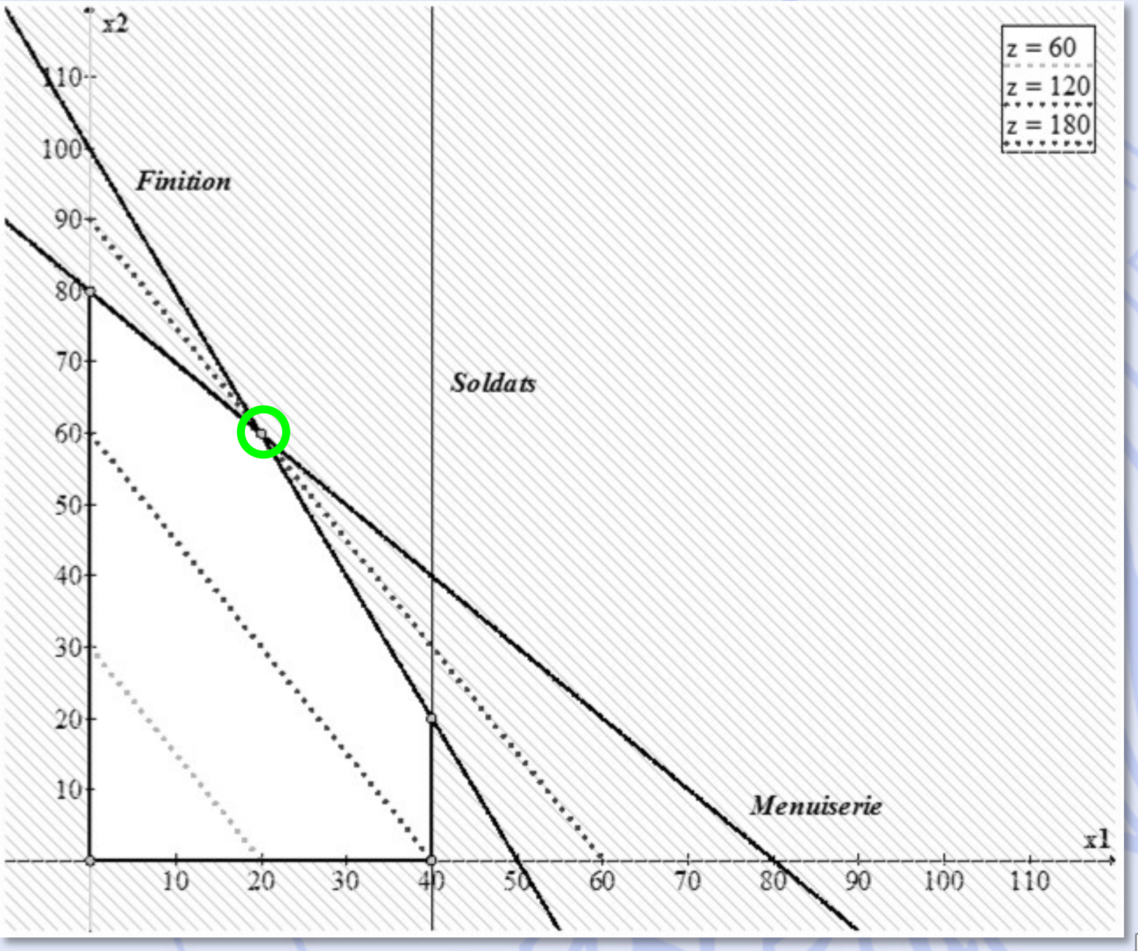
Solu



- Contraintes
- Courbes iso-profit
- Solution optimale

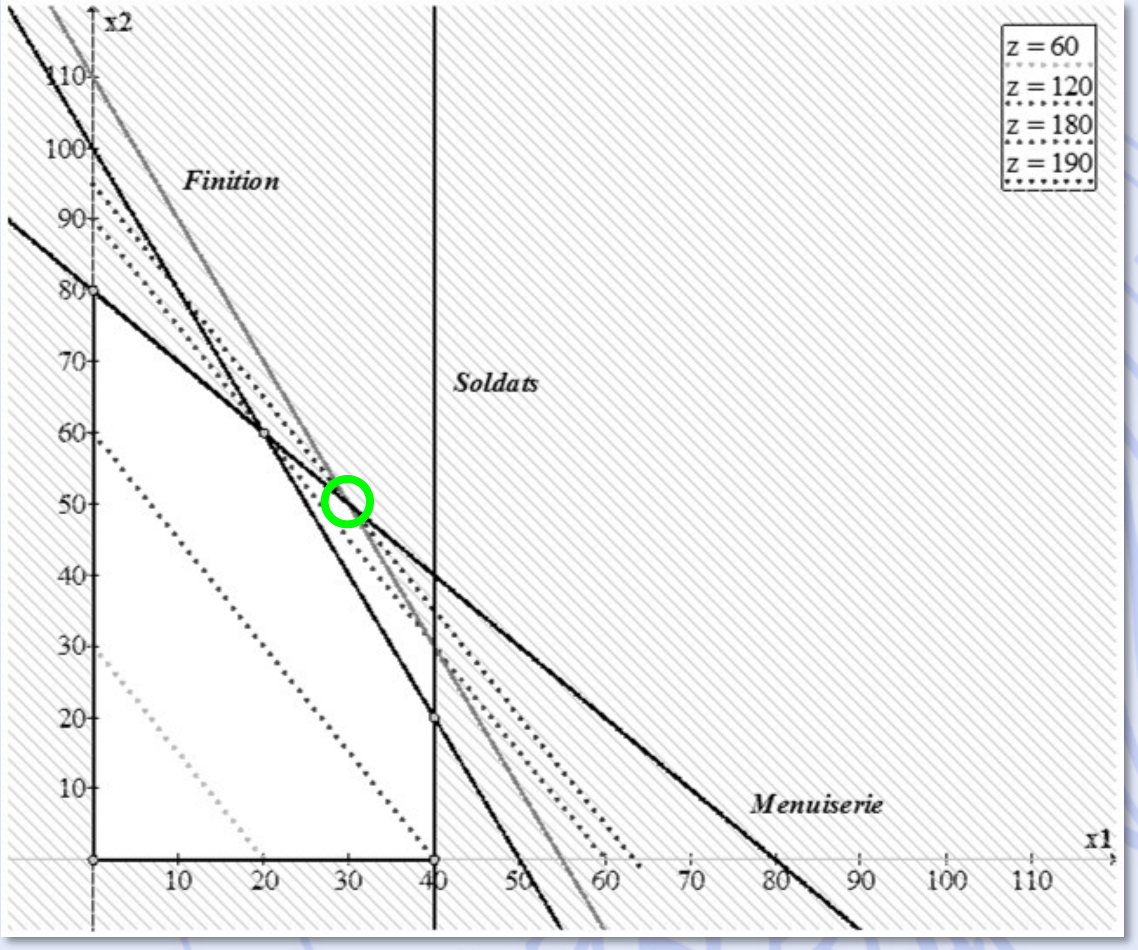
Contrainte active - Finition

100 heures



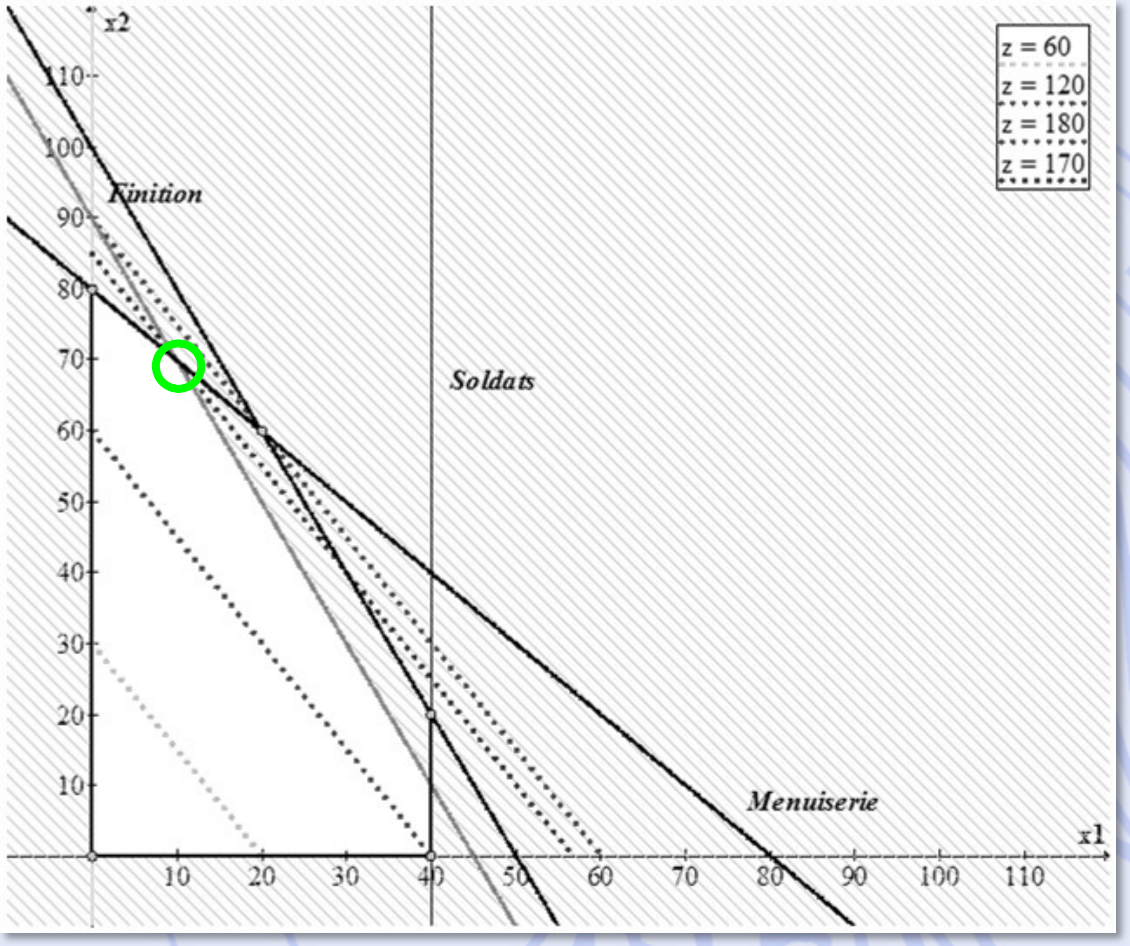
Contrainte active - Finition

110 heures



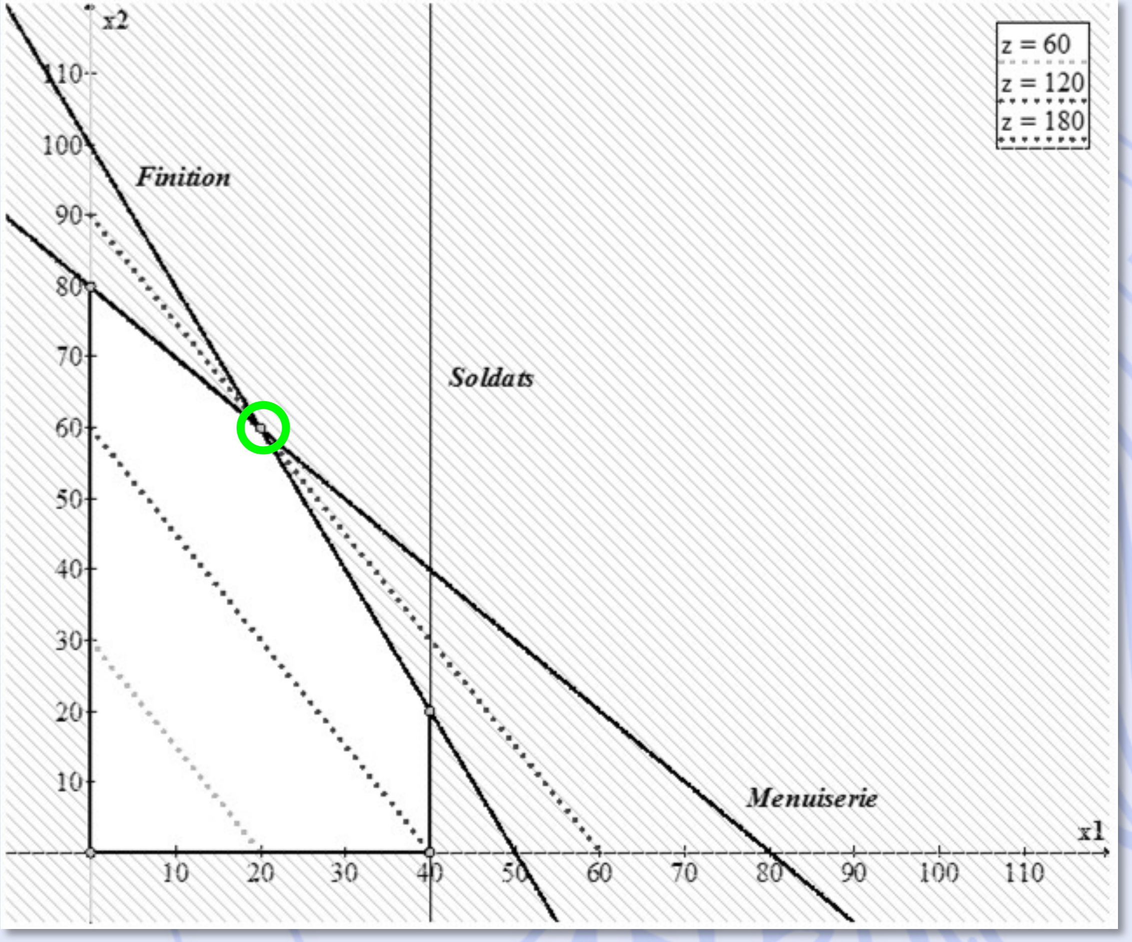
Contrainte active - Finition

90 heures



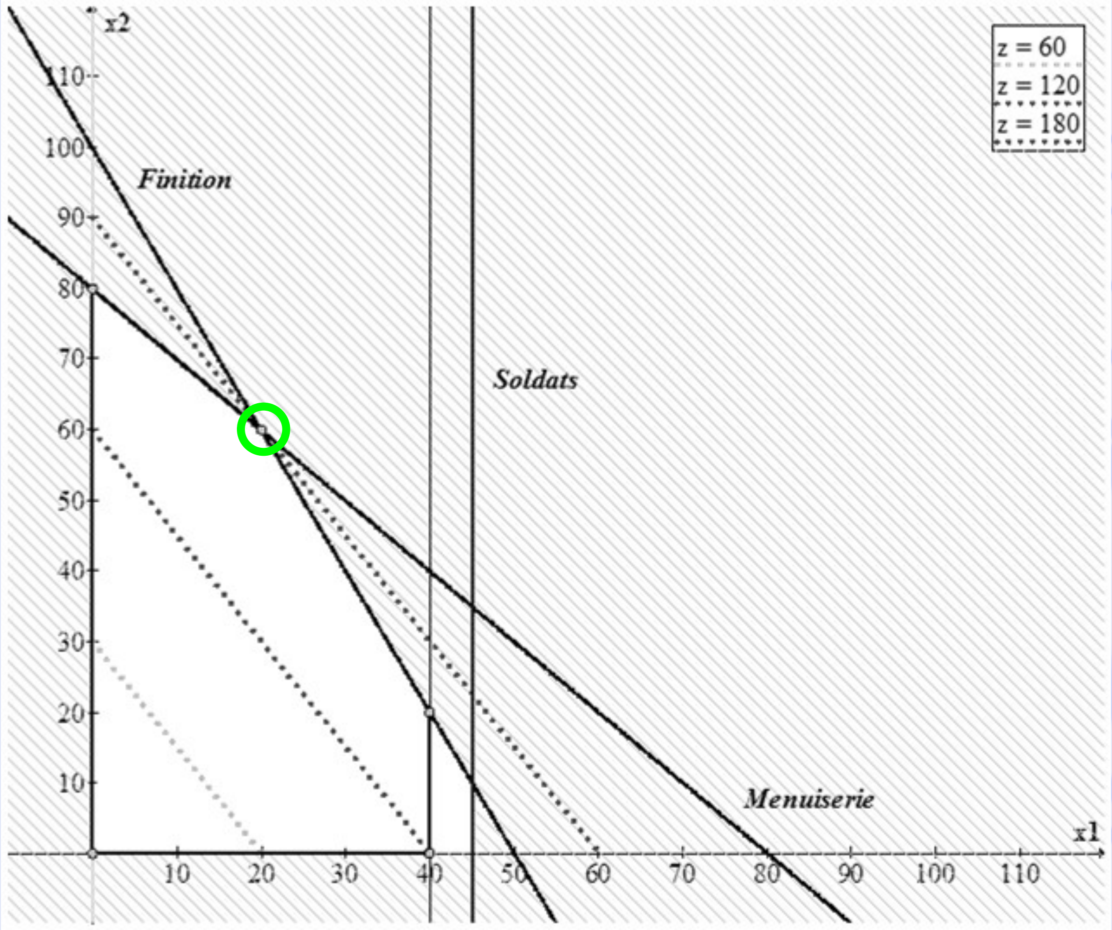
Contrainte non active - Soldats

40 soldats



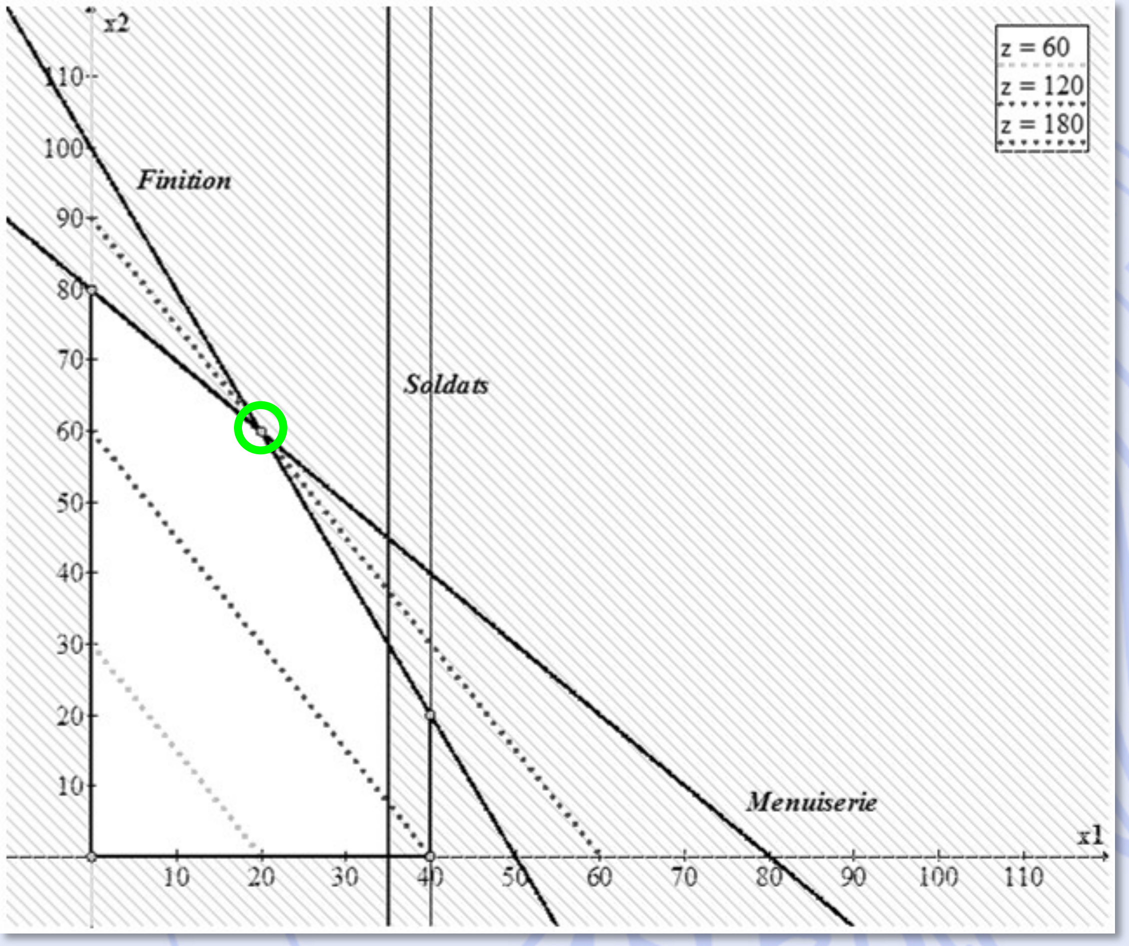
Contrainte non active - Soldats

45 soldats



Contrainte non active - Soldats

35 soldats



Solutions optimales multiples

Une entreprise du secteur automobile fabrique des voitures et des camions. Chaque véhicule doit être traité dans l'atelier de peinture et dans l'atelier de carrosserie. La capacité de l'atelier de peinture permet de traiter 40 camions par jour (si l'on ne peint que des camions), ou 60 voitures par jour (si l'on ne peint que des voitures). De la même façon la capacité de l'atelier de carrosserie est limitée à 50 camions par jour et à 50 voitures par jour. Chaque camion produit rapporte €300, et chaque voiture €200. Déterminer un plan de production quotidien qui permette de maximiser le profit de l'entreprise.

Solutions optimales multiples

x_1 = number of trucks produced daily
x_2 = number of cars produced daily

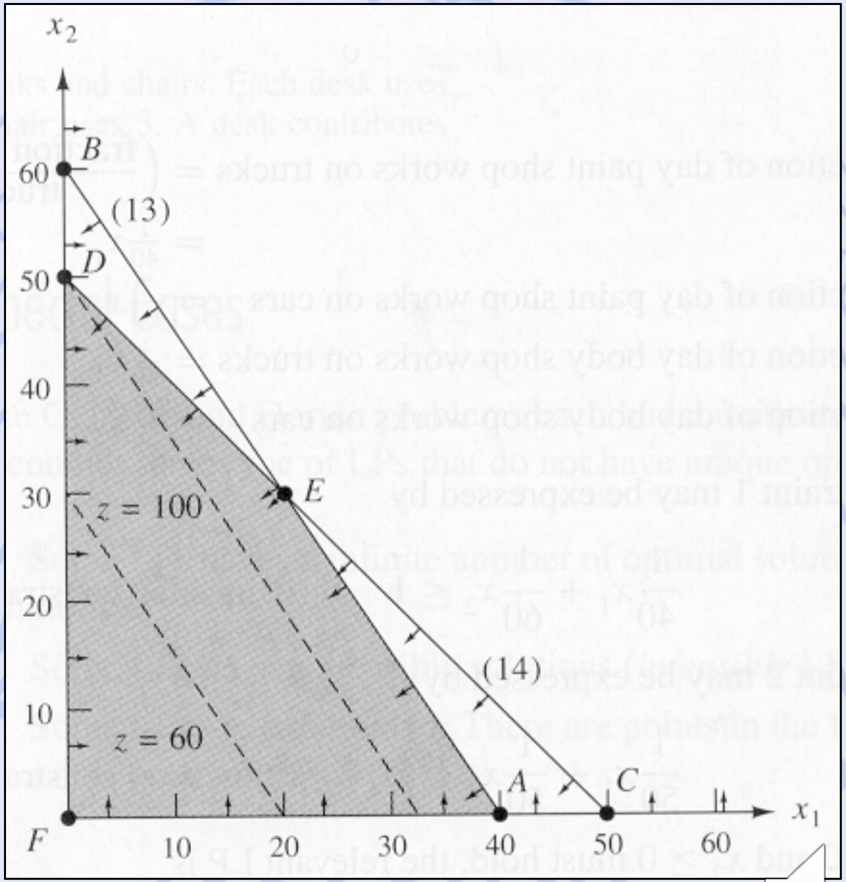
$$\max \ z = 300x_1 + 200x_2$$

s.t.

$$\frac{1}{40}x_1 + \frac{1}{60}x_2 \le 1$$

$$\frac{1}{50}x_1 + \frac{1}{50}x_2 \le 1$$

$$x_1 \ge 0 \qquad x_2 \ge 0$$

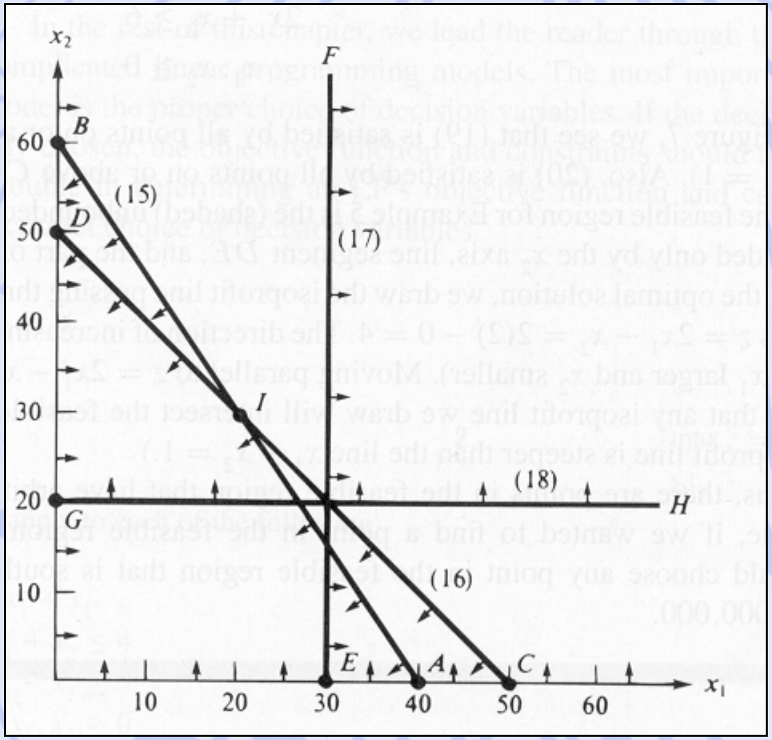


Exemple de PL contradictoire

Suppose now that the auto company is required to product at least 30 trucks and 20 cars per day.

→ 2 additional constraints:

$x_1 \geq 30 \qquad x_2 \geq 20$

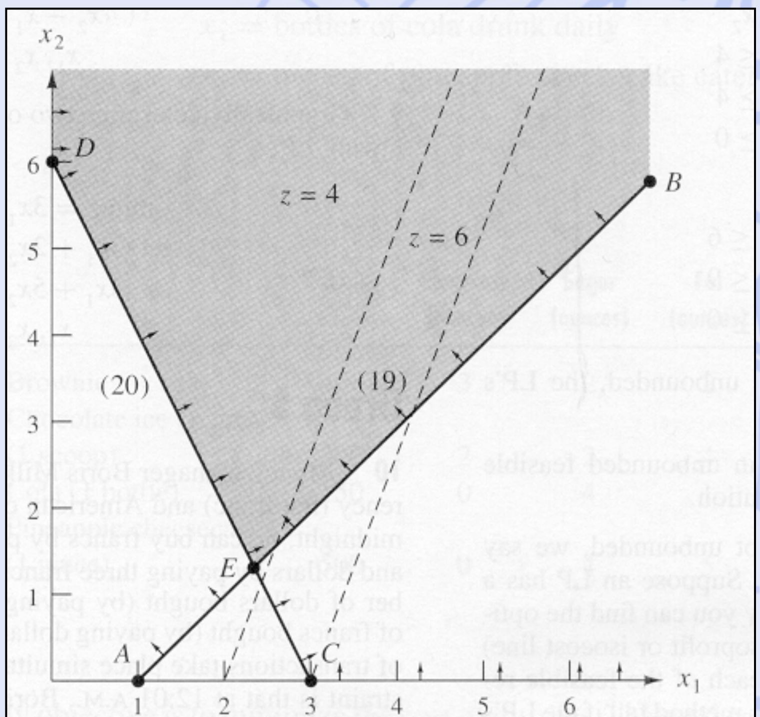


Exemple de PL non borné

$$\max\ z = 2x_1 - x_2$$

s.t.

$$x_1 - x_2 \leq 1$$
$$2x_1 + x_2 \geq 6$$
$$x_1 \geq 0 \qquad x_2 \geq 0$$



Solution optimale

- Peut être :
  - Unique → sommet du domaine réalisable,
  - Multiple → côté du domaine réalisable,
  - Infinie (contraintes manquantes ?),
  - Impossible (contraintes incompatibles !).
- Certaines contraintes sont actives (binding) : LHS = RHS
- Certaines contraintes sont non-actives (nonbinding): LHS ≠ RHS (différence (écart) = slack)

PL typiques : Régime alimentaire (diet)

Four foods are available for consumption: brownies, chocolate ice cream, cola and pineapple cheesecake. One brownie costs ¢50, one scoop of chocolate ice cream costs ¢20, one bottle of cola costs ¢30, and one piece of cheesecake costs ¢80. Each day, you must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar and 8 oz of fat. Formulate a LP to satisfy these requirements at minimum cost.

Per unit:	Calories	Chocolate	Sugar	Fat
Brownie	400	3	2	2
Chocolate ice cream	200	2	2	4
Cola	150	0	4	1
Cheesecake	500	0	4	5

PL typiques : Régime alimentaire

x_1	number of brownies
x_2	number of scoops of chocolate ice cream
x_3	bottles of cola
x_4	pieces of pineapple cheesecake

- LP formulation:

$$\min z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$\begin{aligned} 400x_1 + 200x_2 + 150x_3 + 500x_4 &\geq 500 \\ 3x_1 + 2x_2 &\geq 6 \\ 2x_1 + 2x_2 + 4x_3 + 4x_4 &\geq 10 \\ 2x_1 + 4x_2 + x_3 + 5x_4 &\geq 8 \end{aligned}$$

$$x_i \geq 0 \quad (i = 1, 2, 3, 4)$$

- LP solution:

$$x_1 = 0 \quad x_2 = 3 \quad x_3 = 1 \quad x_4 = 0 \quad z = 90$$

- Slacks:

$$t_1 = 250 \quad t_2 = 0 \quad t_3 = 0 \quad t_4 = 5$$

PL typiques : Horaire de travail

- Un fast-food a besoin d'un nombre différent d'employés plein-temps chaque jour de la semaine :

Jour	Lu	Ma	Me	Je	Ve	Sa	Di
Nb.	17	13	15	19	14	16	11
Coût	100	100	100	100	100	150	200

- Chaque employé travaille 5 jours de suite et a ensuite 2 jours de congé.
- Il y a donc 7 horaires de travail possibles :
  Lu-Ve, Ma-Sa, Me-Di, Je-Lu, Ve-Ma, Sa-Me, Di-Je

PL typiques : Horaire de travail

Horaires	Lu-Ve	Ma-Sa	Me-Di	Je-Lu	Ve-Ma	Sa-Me	Di-Je				
Salaire hebd.	500	550	650	650	650	650	600				
	x1	x2	x3	x4	x5	x6	x7			z	
Nb. Tps-plein	6,33333333	5	0,33333333	7,33333333	0	3,33333333	0		Nb. Total	13066,6667	
Contraintes								LHS	<=, >= or =	RHS	Ecart
Lundi	1			1	1	1	1	17	>=	17	0
Mardi	1	1			1	1	1	14,6666667	>=	13	1,66666667
Mercredi	1	1	1			1	1	15	>=	15	0
Jeudi	1	1	1	1			1	19	>=	19	0
Vendredi	1	1	1	1	1			19	>=	14	5
Samedi		1	1	1	1	1		16	>=	16	0
Dimanche			1	1	1	1	1	11	>=	11	0

- Solution optimale non entière !
- Arrondir ?
- Vers le haut.

PL typiques : Horaire de travail

Horaires	Lu-Ve	Ma-Sa	Me-Di	Je-Lu	Ve-Ma	Sa-Me	Di-Je				
Salaire hebd.	500	550	650	650	650	650	600				
	x1	x2	x3	x4	x5	x6	x7			z	
Nb. Tps-plein	7	5	1	8	0	4	0		Nb. Total	14700	
Contraintes								LHS	<=, >= or =	RHS	Ecart
Lundi	1			1	1	1	1	19	>=	17	2
Mardi	1	1			1	1	1	16	>=	13	3
Mercredi	1	1	1			1	1	17	>=	15	2
Jeudi	1	1	1	1			1	21	>=	19	2
Vendredi	1	1	1	1	1			21	>=	14	7
Samedi		1	1	1	1	1		18	>=	16	2
Dimanche			1	1	1	1	1	13	>=	11	2

- z = 14700 au lieu de 13066,66… !
- Solution optimale entière ?
- Passage de LP à IP (Integer Programming).

PL typiques : Horaire de travail

Horaires	Lu-Ve	Ma-Sa	Me-Di	Je-Lu	Ve-Ma	Sa-Me	Di-Je				
Salaire hebd.	500	550	650	650	650	650	600				
	x1	x2	x3	x4	x5	x6	x7			z	
Nb. Tps-plein	6	6	0	8	0	2	1		Nb. Total	13400	
Contraintes								LHS	<=, >= or =	RHS	Ecart
Lundi	1			1	1	1	1	17	>=	17	0
Mardi	1	1			1	1	1	15	>=	13	2
Mercredi	1	1	1			1	1	15	>=	15	0
Jeudi	1	1	1	1			1	21	>=	19	2
Vendredi	1	1	1	1	1			20	>=	14	6
Samedi		1	1	1	1	1		16	>=	16	0
Dimanche			1	1	1	1	1	11	>=	11	0

- z = 13400 au lieu de 14700 !
- Supérieur à 13066,66…